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On the relation of the stationary Toda equation and the symplectic maps

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Abstract. It is proved that every solution of the stationary Toda equation in the generic case is represented as a discrete orbit of the symplectic map obtained through nonlinearization of the Toda eigenvalue problem. As an application, the calculation of the finite-band solution of the Toda lattice equation is reduced to the solution of a system of ODEs plus a simple iterative process of the symplectic map.

1. Introduction

Although the integrability of differential equations of motion was the ultimate goal in the early years of classical mechanics, there were then only a very few examples of integrable systems; including the famous Jacobi's geodesics on an ellipsoid, Neumann's constrained harmonic oscillator and the integrable cases of top due to Euler, Lagrange and Kovalevskaya. In recent years, quite a few new finite-dimensional integrable systems have been discovered through various reduction techniques from soliton hierarchies (see [1–5]). Nevertheless, it was only in 1991 that the framework of the discrete version of classical integrable systems was found, which is expressed in the language of integrable symplectic maps (see [6, 7]).

The nonlinearization technique [1–3], or the restricted flow technique [4, 5], which is similar, is proved to be a powerful tool for obtaining new finite-dimensional integrable systems from various soliton hierarchies. The aim of the present paper is to show that the technique is also effective, in the discrete case, of finding new integrable symplectic maps [6, 7]. The Toda lattice hierarchy is studied in detail to illustrate the general principles. The symplectic maps in the Bargmann–Garnier and Neumann cases, with their conserved integrals, are obtained in [8, 9]. Here a new understanding is given. It is shown that in the general case each solution of the stationary Toda equation is represented as a discrete orbit of the integrable symplectic map, which is obtained through nonlinearization of the Toda eigenvalue problem. To prove this difficult existence theorem, just as in the continuous case of the Heisenberg hierarchy [10], the following tools are essential and have general significance:

(i) the total difference formulae, which yield the explicit recursive formulae for the Toda gradients $\{g^{(j)}\}$ and the conserved integrals of the symplectic map in a systematic way (see lemma 1.1);

(ii) the commutative formula, which gives the commutative representation of the Toda vector fields (see lemma 3.1); and

(iii) the interpolation formula, which establishes the inner relationship of the Lenard equation and the Lenard eigenvalue problem; the former defines the soliton equations, while the latter determines the nonlinearized eigenvalue problems (see (4.19)).

Direct relations between the integrals $\{F_s\}$ of the symplectic map, the Toda vector fields $\{X^{(s)}\}$ and the time parts $\{V^{(s)}\}$ of the Lax pair are revealed in section 6.

As an application, the calculation of the finite-band solution of the Toda equation is decomposed into two steps: the solution of a Hamiltonian system of ODEs plus an iterative process of the symplectic map. A numerical example shows the discrete evolution picture of the solutions.

A P Fordy asked for an explanation of the integrability mechanism of the nonlinearized eigenvalue problem (NEP). It turns out that there exists some kind of equivalence between the integrability of the stationary soliton equations and the NEP. A rigorous treatment is made in this respect for the Heisenberg case as a continuous model [10], while the present paper provides a discrete example.

2. The Toda vector fields

Let E be the shift operator: $Ef(n) = f(n+1)$, $E^{-1}f(n) = f(n-1)$ and $\Delta = E - 1$, $\Delta^- = 1 - E^{-1}$. Consider the Toda eigenvalue problem:

$$L\psi \equiv a_{n-1}\psi_{n-1} + b_n\psi_n + a_n\psi_{n+1} = \lambda\psi_n \qquad \text{or} \qquad L\psi \equiv (E^{-1}a + b + aE)\psi = \lambda\psi. \tag{2.1}$$

It is well known that

$$\nabla\lambda = \begin{pmatrix} \delta\lambda/\delta a_n \\ \delta\lambda/\delta b_n \end{pmatrix} = \begin{pmatrix} 2\psi_n\psi_{n+1} \\ \psi_n^2 \end{pmatrix} \tag{2.2}$$

$$(K - \lambda J)\nabla\lambda = 0 \tag{2.3}$$

where K, J are the Lenard pair of operators:

$$K = \begin{pmatrix} \frac{1}{2}a(\Delta + \Delta^-)a & a\Delta b \\ b\Delta^- a & 2(a^2\Delta + \Delta^- a^2) \end{pmatrix}$$

$$J = \begin{pmatrix} 0 & a\Delta \\ \Delta^- a & 0 \end{pmatrix}. \tag{2.4}$$

Consider the Lenard recursive equations:

$$J\xi^{(-1)} = 0 \qquad K\xi^{(j-1)} = J\xi^{(j)}. \tag{2.5}$$

The former has two linearly independent solutions:

$$g_n^{(-2)} = \begin{pmatrix} a_n^{-1} \\ 0 \end{pmatrix} \qquad g_n^{(-1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.6a}$$

while the latter has special polynomial solutions:

$$g_n^{(0)} = \begin{pmatrix} 2a_n \\ b_n \end{pmatrix} \qquad g_n^{(1)} = \begin{pmatrix} 2a_n(b_n + b_{n+1}) \\ a_{n-1}^2 + a_n^2 + b_n^2 \end{pmatrix}$$
$$g_n^{(2)} = \begin{pmatrix} 2a_n(a_{n-1}^2 + a_n^2 + a_{n+1}^2 + b_n^2 + b_n b_{n+1} + b_{n+1}^2) \\ a_{n-1}^2(b_{n-1} + 2b_n) + a_n^2(2b_n + b_{n+1}) + b_n^3 \end{pmatrix} \qquad \text{etc.} \tag{2.6b}$$

$g^{(-2)}$ has a special position, since $Kg^{(-2)} = Jg^{(-2)} = 0$, and plays a special role in the whole structure. The first few Toda vector fields $X^{(j)} = Jg^{(j)}$ are

$$X_n^{(-1)} = 0 \qquad X_n^{(0)} = \begin{pmatrix} a_n(b_{n+1} - b_n) \\ 2(a_n^2 - a_{n-1}^2) \end{pmatrix}$$
$$X_n^{(1)} = \begin{pmatrix} a_n(a_{n+1}^2 - a_{n-1}^2 + b_n^2 - b_{n-1}^2) \\ 2a_n^2(b_{n+1} + b_n) - 2a_{n-1}^2(b_n + b_{n-1}) \end{pmatrix}. \tag{2.7}$$

The general solution of (2.5) is expressed as the linear combination

$$\xi^{(j)} = c_0 g^{(j)} + c_1 g^{(j-1)} + \cdots + c_{j+1} g^{(-1)} + \delta_{j+2} g^{(-2)}. \tag{2.8}$$

Lemma 1.1 (Total difference formula). For any functions α, β of discrete variables $n \in \mathbb{Z}$

(i)

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \cdot (K - \lambda J) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \Delta \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \cdot T \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\}$$
$$= \Delta \{ \tfrac{1}{2} a_{n-1} a_n \alpha_{n-1} \alpha_n + (b_n - \lambda) a_{n-1} \alpha_{n-1} \beta_n + 2a_{n-1}^2 \beta_{n-1} \beta_n \} \tag{2.9}$$

(ii)
$$g^{(-2)} \cdot (K - \lambda J) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \Delta \left\{ S \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\}$$
$$= \Delta \{ \tfrac{1}{2}(a_n \alpha_n + a_{n-1} \alpha_{n-1}) + (b_n - \lambda) \beta_n \} \tag{2.10}$$

where $T = T_1 - \lambda T_0$, $S = S_1 - \lambda S_0$ with

$$T_1 = \begin{pmatrix} \tfrac{1}{2} a E^{-1} a & 0 \\ b E^{-1} a & 2E^{-1} a^2 \end{pmatrix} \qquad T_0 = \begin{pmatrix} 0 & 0 \\ E^{-1} a & 0 \end{pmatrix} \tag{2.11}$$

$$S_1 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \tfrac{1}{2}(1 + E^{-1}) a\alpha + b\beta \qquad S_0 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \beta. \tag{2.12}$$

Proof. By direct calculations, resorting to the discrete Leibnitz rule, $f\Delta g + g\Delta^- f = \Delta(gE^{-1}f)$. □

Proposition 1.2. The polynomial solution $g_n^{(j)} = (g_{1,n}^{(j)}, g_{2,n}^{(j)})^T$ satisfies the explicit recursive formula

$$g_{1,n}^{(s)} = b_{n+1} g_{1,n}^{(s-1)} + 2a_n (g_{2,n+1}^{(s-1)} + g_{2,n}^{(s-1)}) - \sum_{j+k=s-1} g_{1,n}^{(j)} g_{2,n+1}^{(k)}$$
$$+ \sum_{j+k=s-2} \{ g_{1,n}^{(j)} (\tfrac{1}{2} a_{n+1} g_{1,n+1}^{(k)} + b_{n+1} g_{2,n+1}^{(k)}) + 2a_n g_{2,n}^{(j)} g_{2,n+1}^{(k)} \} \qquad s = 2, 3, 4, \ldots \tag{2.13}$$

$$g_{2,n}^{(s)} = \tfrac{1}{2}(a_n g_{1,n}^{(s-1)} + a_{n-1} g_{1,n-1}^{(s-1)}) + b_n g_{2,n}^{(s-1)} \qquad s = 0, 1, 2, \ldots.$$

Proof. Put $\{g^{(j)}\}$ on the carrier of the Laurent series

$$g = g^{(-1)} + \sum_{s=0}^{\infty} \frac{1}{\lambda^{s+1}} g^{(s)}$$

which satisfies $(K - \lambda J)g = 0$. Hence, we have two quantities independent of $n \in \mathbb{Z}$

$$g \cdot Tg = \text{constant} = 0 \qquad Sg = \text{constant} = -\lambda$$

and the required (2.13) is obtained by comparing the coefficients of the same power of λ. □

Remark 1.3. As by-products, we have

$$S_0 g^{(-1)} = 1 \qquad S_1 g^{(j-1)} = S_0 g^{(j)} \qquad j = 0, 1, 2, \ldots. \tag{2.14}$$

3. The commutative representation

By introducing $p_n = a_{n-1}\psi_{n-1}$, $q_n = \psi_n$, (2.1) is transformed into

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = U_n \begin{pmatrix} p_n \\ q_n \end{pmatrix} \tag{3.1}$$

with

$$U_n = \begin{pmatrix} 0 & a_n \\ -\dfrac{1}{a_n} & -\dfrac{b_n - \lambda}{a_n} \end{pmatrix} = U \begin{pmatrix} a_n \\ b_n \end{pmatrix}. \tag{3.2}$$

The discrete zero-curvature equation

$$U_{nt} = V_{n+1} U_n - U_n V_n \tag{3.3}$$

is defined as the compatibility condition of the Lax pair

$$y_{n+1} = U_n y_n \qquad y_{nt} = V_n y_n.$$

Let $u = (a, b)^T$ and

$$U_*(u)[\dot{u}] = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} U(u + \varepsilon \dot{u}) = \begin{pmatrix} 0 & \dot{a} \\ \dfrac{1}{a^2}\dot{a} & \dfrac{b-\lambda}{a^2}\dot{a} - \dfrac{1}{a}\dot{b} \end{pmatrix}. \tag{3.4}$$

A direct calculation gives the following lemma.

Lemma 3.1 (Commutative formula). For arbitrary function $G_n = (\alpha_n, \beta_n)^T$, of discrete variables $n \in \mathbb{Z}$,

$$V_{n+1}U_n - U_nV_n = U_* \begin{pmatrix} a_n \\ b_n \end{pmatrix} [(K - \lambda J)G_n] \tag{3.5}$$

where the zero trace V is defined as

$$V_n = \sigma(G_n) = \begin{pmatrix} \frac{1}{2}(a_n\alpha_n - a_{n-1}\alpha_{n-1}) + (b_n - \lambda)\beta_n & 2a_{n-1}^2\beta_{n-1} \\ -2\beta_n & * \end{pmatrix}. \tag{3.6}$$

Consider a special choice of G:

$$G^{(N)}(\lambda, \xi) = \sum_{j=0}^{N} \xi^{(j-1)}\lambda^{N-j}. \tag{3.7}$$

We have

$$\begin{aligned}(K - \lambda J)G^{(N)} &= K\xi^{(N-1)} + \sum_{j=0}^{N-1}(K\xi^{(j-1)} - J\xi^{(j)})\lambda^{N-j} - (J\xi^{(-1)})\lambda^{N+1} \\ &= J\xi^{(N)} = c_0X^{(N)} + c_1X^{(N-1)} + \cdots + c_NX^{(0)}\end{aligned}$$

Theorem 3.2. Let $G^{(N)}$ be defined in (3.7) and $V^{(N)} = \sigma(G^{(N)})$. Then

(i) $$(K - \lambda J)G^{(N)} = J\xi^{(N)}. \tag{3.8}$$

(ii) $$V_{n+1}^{(N)}U_n - U_nV_n^{(N)} = U_* \begin{pmatrix} a_n \\ b_n \end{pmatrix} [J\xi^{(N)}] \tag{3.9}$$

Corollary 3.3. The Toda lattice equation

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix}_t = J\xi^{(N)}$$

is equivalent to the discrete zero-curvature equation

$$U_{nt} = V_{n+1}^{(N)}U_n - U_nV_n^{(N)}.$$

4. The Bargmann–Garnier coordinates for the stationary Toda equation

Let $c_0 = 1$ and $c_1, \ldots, c_N$ be any given constants. Consider the stationary Toda (ST) equation

$$\widetilde{X}^{(N)} = J\xi^{(N)} = X^{(N)} + c_1X^{(N-1)} + \cdots + c_NX^{(0)} = 0. \tag{4.1}$$

Define $\xi^{(-1)} = g^{(-1)}$ and

$$\begin{aligned}\xi^{(j)} &= g^{(j)} + c_1g^{(j-1)} + \cdots + c_{j+1}g^{(-1)} + \delta_{j+2}g^{(-2)} \\ &= \tilde{\xi}^{(j)} + \delta_{j+2}g^{(-2)}.\end{aligned} \tag{4.1a}$$

$j = 0, 1, 2, \ldots$. Then (3.7) is decomposed into

$$G^{(N)}(\lambda, \xi) = G^{(N)}(\lambda, \tilde{\xi}) + \delta(\lambda) g^{(-2)} \tag{4.2}$$

with

$$\delta(\lambda) = \delta_2 \lambda^{N-1} + \delta_3 \lambda^{N-2} + \cdots + \delta_{N+1}. \tag{4.2a}$$

Since $Kg^{(-2)} = Jg^{(-2)} = 0$, we have freedom in the choice of $\{\delta_j\}$. According to theorem 3.2, (4.1) has two equivalent forms:

(i) $$(K - \lambda J) G^{(N)} = 0 \tag{4.3}$$

(ii) $$V_{n+1}^{(N)} U_n - U_n V_n^{(N)} = 0. \tag{4.4}$$

Proposition 4.1. The discrete flow defined by (4.1) has three conserved quantities (independent of $n \in \mathbb{Z}$), expressed in terms of $G_n^{(N)} = (\alpha_n, \beta_n)^T$:

(i) $$\det V^{(N)} = 4a_{n-1}^2 \beta_{n-1} \beta_n - [\tfrac{1}{2} a_n \alpha_n - a_{n-1} \alpha_{n-1}) + (b_n - \lambda) \beta_n]^2 \tag{4.5}$$

(ii) $$G^{(N)} \cdot TG^{(N)} = \tfrac{1}{2} a_{n-1} a_n \alpha_{n-1} \alpha_n + (b_n - \lambda) a_{n-1} \alpha_{n-1} \beta_n + 2a_{n-1}^2 \beta_{n-1} \beta_n \tag{4.6}$$

(iii) $$D^{(N)} \equiv SG^{(N)} = \tfrac{1}{2}(a_n \alpha_n + a_{n-1} \alpha_{n-1}) + (b_n - \lambda) \beta_n \tag{4.7}$$

with the relations

$$\det V^{(N)} - 2G^{(N)} \cdot TG^{(N)} = -[D^{(N)}]^2 \tag{4.8}$$

$$G^{(N)} \cdot TG^{(N)} - a_{n-1} \alpha_{n-1} D^{(N)} = \tfrac{1}{2} a_{n-1}^2 (4\beta_{n-1} \beta_n - \alpha_{n-1}^2). \tag{4.9}$$

Proof. We have

$$V_{n+1}^{(N)} = U_n V_n^{(N)} U_n^{-1}$$

by (4.4). Hence, (4.5) is independent of n and (4.6) and (4.7) are conserved integrals of (4.3) according to the total difference formulae (2.9) and (2.10). Direct calculations yield (4.8) and (4.9). □

Lemma 4.2. $\det V^{(N)}$ is independent of $\delta(\lambda)$ and

$$\det V^{(N)} = -(\lambda - \lambda_1) \cdots (\lambda - \lambda_{2N+2}). \tag{4.10}$$

Proof. Because the components of $G^{(N)}(\lambda, \xi)$ are

$$\alpha_n = \tilde{\alpha}_n + a_n^{-1} \delta(\lambda) = (2a_n + \delta_2 a_n^{-1}) \lambda^{N-1} + \cdots$$
$$\beta_n = \tilde{\beta}_n = \lambda^N + \cdots$$

by (4.2), we see that α, β in (4.5) can be replaced by $\tilde{\alpha}, \tilde{\beta}$ and the highest term in (4.5) is $-\lambda^{2N+2}$. □

Consider the generic case: there exist N distinct elements among $\lambda_1, \dots, \lambda_{2N+2}$. Without loss of generality, let them be $\lambda_1, \dots, \lambda_N$. Define

$$P(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_N) = \sum_{j=0}^{N} l_j \lambda^{N-j}. \tag{4.11}$$

Lemma 4.3. Let $\widetilde{D}^{(N)} = S[G^{(N)}(\lambda, \tilde{\xi})]$. Then

(i) $$D^{(N)} = \widetilde{D}^{(N)} + \delta(\lambda) \tag{4.12}$$

(ii) $$\widetilde{D}^{(N)} = -(\lambda^{N+1} + c_1\lambda^N + \cdots + c_N\lambda) + S_1\tilde{\xi}^{(N-1)}. \tag{4.13}$$

Proof.

$$D^{(N)} = S[\widetilde{G}^{(N)} + g^{(-2)}\delta(\lambda)] = S[\widetilde{G}^{(N)}] + S[g^{(-2)}]\delta(\lambda) = \widetilde{D}^{(N)} + \delta(\lambda)$$

$$\begin{aligned}\widetilde{D}^{(N)} &= (S_1 - \lambda S_0)\sum_{j=0}^{N} \tilde{\xi}^{(j-1)}\lambda^{N-j} \\ &= S_1\tilde{\xi}^{(N-1)} + \sum_{j=0}^{N-1}(S_1\tilde{\xi}^{(j-1)} - S_0\tilde{\xi}^{(j)})\lambda^{N-j} - S_0\tilde{\xi}^{(-1)}\lambda^{N-1}\end{aligned}$$

which gives (4.13) since $S_0 g^{(-1)} = 1$ and

$$\begin{aligned}S_1\tilde{\xi}^{(j-1)} - S_0\tilde{\xi}^{(j)} &= S_1(g^{(j-1)} + c_1 g^{(j-2)} + \cdots + c_j g^{(-1)}) \\ &\quad - S_0(g^{(j)} + c_1 g^{(j-1)} + \cdots + c_j g^{(0)} + c_{j+1} g^{(-1)}) \\ &= -c_{j+1} S_0 g^{(-1)} = -c_{j+1}\end{aligned}$$

where (2.14) is used.

Now a crucial expression is to establish $D^{(N)}$ by fully using the freedom of choice of $\delta(\lambda)$. Let

$$\gamma = c_1 - l_1 = c_1 + \sum_{k=1}^{N} \lambda_k. \tag{4.14}$$

Then

$$(\lambda + \gamma)P(\lambda) = \lambda^{N+1} + c_1\lambda^N + \cdots.$$

By (4.13), $\widetilde{D}^{(N)} + (\lambda + \gamma)P(\lambda)$ is a polynomial with degree not greater than $N - 1$, and will be taken as $-\delta(\lambda)$. Hence,

$$D^{(N)} = \widetilde{D}^{(N)} + \delta(\lambda) = -(\lambda + \gamma)P(\lambda)$$

and

$$\begin{aligned}2G^{(N)} \cdot TG^{(N)} &= \det V^{(N)} + [D^{(N)}]^2 \\ &= P(\lambda)\{-(\lambda - \lambda_{N+1}) \cdots (\lambda - \lambda_{2N+2}) + (\lambda + \gamma)^2 P(\lambda)\}.\end{aligned}$$

A direct calculation gives

$$G^{(N)} \cdot TG^{(N)} = -\delta_2 \lambda^{2N} + \cdots .$$

Therefore,

$$\begin{aligned} G^{(N)} \cdot TG^{(N)} &= -\delta_2 P(\lambda) R(\lambda) \\ &= -\delta_2 (\lambda - \lambda_1) \cdots (\lambda - \lambda_N)(\lambda - \mu_1) \cdots (\lambda - \mu_N). \end{aligned} \qquad \square$$

Proposition 4.4. The conserved integrals of (4.1) are polynomials of λ along the discrete flows

$$\det V^{(N)} = -Q(\lambda) P(\lambda) \tag{4.15}$$

$$G^{(N)} \cdot TG^{(N)} = -\delta_2 R(\lambda) P(\lambda) \tag{4.16}$$

$$D^{(N)} = SG^{(N)} = -(\lambda + \gamma) P(\lambda) \tag{4.17}$$

with a common factor $P(\lambda)$ and the relation

$$Q(\lambda) = (\lambda + \gamma)^2 P(\lambda) + 2\delta_2 R(\lambda). \tag{4.18}$$

4.1. Construction of the canonical coordinates

Decompose $G^{(N)}$ and introduce $\{\Gamma^{(j)}\}$ by the 'interpolation formula'

$$\begin{aligned} G^{(N)}(\lambda, \xi) &= \xi^{(-1)} P(\lambda) + \sum_{j=1}^{N} \hat{\xi}^{(j-1)} \lambda^{N-j} \\ &= \xi^{(-1)} P(\lambda) + \sum_{j=1}^{N} \frac{P(\lambda)}{\lambda - \lambda_j} \Gamma^{(j)} \end{aligned} \tag{4.19}$$

where $\hat{\xi}^{(j-1)} = \xi^{(j-1)} - l_j \xi^{(-1)}$, which determines a linear automorphism

$$(\hat{\xi}^{(0)}, \ldots, \hat{\xi}^{(N-1)}) \to (\Gamma^{(1)}, \ldots, \Gamma^{(N)})$$

with

$$\begin{cases} G^{(N)}(\lambda_k, \xi) = P'(\lambda_k) \Gamma^{(k)} \qquad k = 1, \ldots, N & (4.20a) \\ \displaystyle\sum_{j=1}^{N} \Gamma^{(j)} = \hat{\xi}^{(0)} = \begin{pmatrix} 2a_n + \delta_2 a_n^{-1} \\ b_n + \gamma. \end{pmatrix} & (4.20b) \end{cases}$$

By (4.3), with $\lambda = \lambda_k$,

$$(K - \lambda_k J) \Gamma^{(k)} = 0 \qquad k = 1, \ldots, N. \tag{4.21}$$

The left-hand side of (4.9) has a common factor $P(\lambda)$ by theorem 4.4. Therefore, by putting $\lambda = \lambda_k$ we have

$$0 = 4\beta_{n-1}\beta_n - \alpha_{n-1}^2 = 4\Gamma_{2,n-1}^{(k)} \Gamma_{2,n}^{(k)} - (\Gamma_{1,n-1}^{(k)})^2$$

which implies

$$\Gamma_n^{(k)} = \begin{pmatrix} \Gamma_{1,n}^{(k)} \\ \Gamma_{2,n}^{(k)} \end{pmatrix} = \begin{pmatrix} 2\psi_n^{(k)}\psi_{n+1}^{(k)} \\ [\psi_n^{(k)}]^2 \end{pmatrix} \tag{4.22}$$

by introducing

$$\psi_n^{(k)} = \sqrt{\Gamma_{2,n}^{(k)}}. \tag{4.23}$$

Theorem 4.5. $\psi^{(k)}$ defined by (4.23) satisfies the Toda eigenvalue problem with constraint conditions

$$\begin{cases} (L-\lambda_k)\psi^{(k)} \equiv a_n\psi_{n+1}^{(k)} + (b_n-\lambda_k)\psi_n^{(k)} + a_{n-1}\psi_{n-1}^{(k)} = 0 & (4.24a) \\ \begin{pmatrix} 2a_n + \delta_2 a_n^{-1} \\ b_n + \gamma \end{pmatrix} = \begin{pmatrix} 2\langle \Psi_n, \Psi_{n+1}\rangle \\ \langle \Psi_n, \Psi_n\rangle \end{pmatrix} & (4.24b) \end{cases}$$

where $\Psi_n = (\psi_n^{(1)}, \ldots, \psi_n^{(N)})^T$, $\langle \eta, \zeta\rangle = \Sigma_1^N \eta^{(j)}\zeta^{(j)}$.

Proof. At $\lambda = \lambda_k$,

$$0 = D^{(N)} = P'(\lambda_k)\psi_n^{(k)}\{(L-\lambda_k)\psi^{(k)}\}.$$

Introduce the Bargmann–Garnier coordinates

$$q_n = \psi_n \qquad p_n = a_{n-1}\psi_{n-1}. \tag{4.25}$$

Then (4.24) implies

$$\begin{pmatrix} a \\ b \end{pmatrix} = f(p,q) \equiv \begin{pmatrix} \pm\sqrt{\langle \Lambda q, q\rangle + \gamma\langle q,q\rangle - \langle q,q\rangle^2 - \langle p,q\rangle - \frac{1}{2}\delta_2} \\ \langle q,q\rangle - \gamma \end{pmatrix} \tag{4.26}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. The Toda eigenvalue problem (4.24*a*) is nonlinearized to be

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = H \begin{pmatrix} p_n \\ q_n \end{pmatrix} \tag{4.27}$$

where

$$H\colon \begin{pmatrix} p \\ q \end{pmatrix} \to \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} aq \\ a^{-1}(\Lambda q + \gamma q - \langle q,q\rangle q - p) \end{pmatrix} \tag{4.28}$$

is a symplectic map, since, by direct calculations,

$$\mathrm{d}p' \wedge \mathrm{d}q' = \mathrm{d}p \wedge \mathrm{d}q. \qquad \square$$

Thus we have the following theorem.

Theorem 4.6. In the generic case, given the constants $c_1, \ldots, c_n$, each solution of the stationary Toda equation (4.1) can be represented as

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = f(p_n, q_n) \tag{4.29}$$

where (p_n, q_n) is a discrete orbit of the symplectic map H

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = H^n \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \tag{4.30}$$

starting from some initial point (p_0, q_0).

Theorem 4.7. Given distinct $\lambda_1, \ldots, \lambda_N$, every discrete orbit $(p_n, q_n)^T$ of the symplectic map H is mapped by f into the solution of the stationary Toda equation (4.1) with some constant coefficients $c_1, \ldots, c_N$.

Proof. The direct verification is more easy.

Denote the space of orbits generated by H by $N(\lambda_1, \ldots, \lambda_N)$, and the solution variety of (4.1) by $M(c_1, \ldots, c_N)$. Consider the union

$$N_N = \bigcup N(\lambda_1, \ldots, \lambda_N)$$

with $(\lambda_1, \ldots, \lambda_N)$ running over all points of C^N with distinct $\lambda_1, \ldots, \lambda_n$, and

$$M_N = \bigcup M(c_1, \ldots, c_N)$$

with $(c_1, \ldots, c_N)$ *running over all generic cases. Then the facts obtained in this section* mean that

$$f\colon N_N \to M_N$$

is an onto mapping. Therefore, the stationary Toda equation (4.1) is equivalent to the symplectic map H in (4.28) in the above sense.

5. Integrability of the symplectic map H

The conserved integrals of H are given in Proposition 4.1 and 4.4. To show the involutivity of the integrals they should be written in terms of the canonical coordinates (p, q). First we have

$$\frac{G^{(N)}(\lambda, \xi)}{P(\lambda)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} 2\psi_n^{(j)}\psi_{n+1}^{(j)} \\ [\psi_n^{(j)}]^2 \end{pmatrix} = \begin{pmatrix} 2Q_\lambda(\Psi_n, \Psi_{n+1}) \\ 1 + Q_\lambda(\Psi_n, \Psi_n) \end{pmatrix} \tag{5.1}$$

where

$$\begin{aligned} Q_\lambda(\eta, \zeta) &= \langle(\lambda - \Lambda)^{-1}\eta, \zeta\rangle \\ &= \sum_{j=1}^{N} \frac{\eta^{(j)}\zeta^{(j)}}{\lambda - \lambda_j} = \sum_{s=0}^{\infty} \frac{1}{\lambda^{s+1}} \langle\Lambda^s \eta, \zeta\rangle. \end{aligned} \tag{5.1a}$$

Define the generating function of the conserved integrals

$$\begin{aligned}\mathcal{F}_\lambda &\equiv \frac{-1}{2P^2(\lambda)} G^{(N)} \cdot T G^{(N)} \\ &= -a_{n-1} a_n Q_\lambda(\Psi_{n-1}, \Psi_n) Q_\lambda(\Psi_n, \Psi_{n+1}) \\ &\quad - (b_n - \lambda) a_{n-1} Q_\lambda(\Psi_{n-1}, \Psi_n)[1 + Q_\lambda(\Psi_n, \Psi_n)] \\ &\quad - a_{n-1}^2 [1 + Q_\lambda(\Psi_{n-1}, \Psi_{n-1})[1 + Q_\lambda(\Psi_n, \Psi_n)]\end{aligned}$$

which is reduced to

$$\begin{aligned}\mathcal{F}_\lambda(p,q) = Q_\lambda(\Lambda p, q) + \gamma Q_\lambda(p,q) - \langle p,q\rangle Q_\lambda(q,q) - Q_\lambda(p,p) + \frac{\delta_2}{2}[1 + Q_\lambda(q,q)] \\ - \begin{vmatrix} Q_\lambda(p,p) & Q_\lambda(p,q) \\ Q_\lambda(q,p) & Q_\lambda(q,q) \end{vmatrix} \qquad (5.2)\end{aligned}$$

due to (4.24). According to (4.16), we have

$$\mathcal{F}_\lambda(p,q) = \frac{\delta_2 R(\lambda)}{2P(\lambda)} = \frac{\delta_2(\lambda - \mu_1)\cdots(\lambda - \mu_N)}{2(\lambda - \lambda_1)\cdots(\lambda - \lambda_N)} \qquad (5.3)$$

along the discrete orbits of H. A direct calculation shows the involutivity

$$\{\mathcal{F}_\lambda, \mathcal{F}_\mu\} = 0 \qquad \forall \lambda, \mu \in C \qquad (5.4)$$

where $\{\cdot,\cdot\}$ stands for the Poisson bracket in $(\mathbb{R}^{2N}, \mathrm{d}p \wedge \mathrm{d}q)$. The Laurent expansion

$$\mathcal{F}_\lambda(p,q) = \frac{\delta_2}{2} + \sum_{s=0}^{\infty} \frac{F_s(p,q)}{\lambda^{s+1}} \qquad (5.5)$$

yields $(s = 1, 2, \ldots)$

$$\begin{aligned}F_0 &= \langle \Lambda p, q\rangle + \gamma\langle p,q\rangle - \langle p,q\rangle\langle q,q\rangle - \langle p,p\rangle + \frac{\delta_2}{2}\langle q,q\rangle \\ F_s &= \langle \Lambda^{s+1} p, q\rangle + \gamma\langle \Lambda^s p,q\rangle - \langle p,q\rangle\langle \Lambda^s q,q\rangle - \langle \Lambda^s p,p\rangle + \frac{\delta_2}{2}\langle \Lambda^s q,q\rangle \\ &\quad + \sum_{j+k=s-1} \begin{vmatrix} \langle \Lambda^j p,p\rangle & \langle \Lambda^j p,q\rangle \\ \langle \Lambda^k q,p\rangle & \langle \Lambda^k q,q\rangle \end{vmatrix}. \qquad (5.6)\end{aligned}$$

Theorem 5.1. The symplectic map H defined by (4.28) is completely integrable with the involutive system of conserved integrals $\{F_j\}$ given by (5.6)

$$\{F_j, F_k\} = 0 \qquad \forall j, k = 0, 1, 2, \ldots. \qquad (5.7)$$

6. The finite-band solution of the Toda lattice equation

It is desirable to find the direct relationship between the integrable symplectic map H and the soliton hierarchy, or more specifically, the relations between $\{F_s\}$, $\{X^{(s)}\}$ and $\{V^{(s)}\}$.

Theorem 6.1. (i) The differential of the map f maps the Hamiltonian vector field $I\nabla F_s$ on N_N into the soliton vector field $X^{(s)}$ on M_N up to a linear combination

$$f_*: I\nabla F_s \rightarrow X^{(s)} + e_1 X^{(s-1)} + \cdots + e_s X^{(0)} = J\eta^{(s)} \tag{6.1}$$

where the gradient $\eta^{(s)}$ is defined by (6.3).

(ii) $I\nabla F_s$ is the result of nonlinearization of the time part $V^{(s)}y$ of the Lax pair:

$$I\nabla_j F_s \equiv \begin{pmatrix} -\partial F_s/\partial q^{(j)} \\ \partial F_s/\partial p^{(j)} \end{pmatrix} = V^{(s)}(\lambda_j, \eta) \begin{pmatrix} p^{(j)} \\ q^{(j)} \end{pmatrix} \tag{6.2}$$

where

$$V^{(s)}(\lambda, \eta) = \sigma(G^{(s)}(\lambda, \eta)).$$

Proof. Let $(p_n, q_n)^T$ be an orbit of H and let $\Psi_n = q_n$. Define

$$\begin{aligned} \eta^{(-1)} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} = g^{(-1)} \\ \eta^{(s)} &= \sum_{j=1}^{N} \lambda_j^s \Gamma^{(j)} = \begin{pmatrix} 2\langle \Lambda^s \Psi_n, \Psi_{n+1} \rangle \\ \langle \Lambda^s \Psi_n, \Psi_n \rangle \end{pmatrix} \qquad s = 0, 1, \ldots . \end{aligned} \tag{6.3}$$

which satisfy the Lenard recursive equations (2.5). Thus,

$$\eta^{(s)} = g^{(s)} + e_1 g^{(s-1)} + \cdots + e_{s+1} g^{(-1)} + \hat{e}_{s+2} g^{(-2)}$$

and

$$J\eta^{(s)} = X^{(s)} + e_1 X^{(s-1)} + \cdots + e_s X^{(0)}.$$

Let $y = (p, q)^T$. Consider the differential of f:

$$\begin{aligned} f_*(y)[\dot{y}] &= \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} f(y + \varepsilon \dot{y}) \\ &= \begin{pmatrix} \frac{1}{2a}\{2\langle \Lambda q, \dot{q}\rangle + 2\gamma\langle q, \dot{q}\rangle - 4\langle q, q\rangle\langle q, \dot{q}\rangle - \langle p, \dot{q}\rangle - \langle q, \dot{p}\rangle\} \\ 2\langle q, \dot{q}\rangle \end{pmatrix} \end{aligned}$$

and the Hamiltonian vector field of F_s

$$\begin{pmatrix} p \\ q \end{pmatrix}^{\cdot} = I\nabla F_s = \begin{pmatrix} -\partial F_s/\partial q \\ \partial F_s/\partial p \end{pmatrix}$$

A direct and tedious calculation gives

$$f_* \begin{pmatrix} p \\ q \end{pmatrix} [I\nabla F_s] = J\eta^{(s)}$$

and (6.2). □

Theorem 6.2. Let $y_0(t_s) = (p_0(t_s), q_0(t_s))^T$ be a solution of the initial-value problem:

$$\frac{\partial}{\partial t_s}\begin{pmatrix} p \\ q \end{pmatrix} = I\nabla F_s \qquad \begin{pmatrix} p \\ q \end{pmatrix}\bigg|_{t_s=0} = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \tag{6.4}$$

and

$$y_n(t_s) = H^n(y_0(t_s)). \tag{6.5}$$

Then $(a_n(t_s), b_n(t_s))^T = f(y_n(t_s))$ solves the Toda lattice equation

$$\frac{\mathrm{d}}{\mathrm{d}t_s}\begin{pmatrix} a_n \\ b_n \end{pmatrix} = J\eta^{(s)} = X^{(s)} + e_1 X^{(s-1)} + \cdots + e_s X^{(0)}. \tag{6.6}$$

Proof. According to (6.2) and theorem 4.6, (6.4) and (6.5) can be put in the form

$$\frac{\mathrm{d}}{\mathrm{d}t_s} y_n = V_n^{(s)} y_n \qquad y_{n+1} = U_n y_n.$$

Therefore,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t_s} U_n - (V_{n+1}U_n - U_n V_n) = U_*\begin{pmatrix} a_n \\ b_n \end{pmatrix}\left[\frac{\mathrm{d}}{\mathrm{d}t_s}\begin{pmatrix} a_n \\ b_n \end{pmatrix} - J\eta^{(s)}\right].$$

By (3.4), U_* is a one-to-one linear map, hence (6.6) is satisfied. □

Consider the special case $s = 0$. Let $(p_0(t), q_0(t))^T$ be the solution of the initial-value problem:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} p \\ q \end{pmatrix} = I\nabla F_0 = \begin{pmatrix} -\Lambda p - \gamma p + \langle q, q\rangle p + 2\langle p, q\rangle q - \delta_2 q \\ \Lambda q + \gamma q - \langle q, q\rangle q - 2p \end{pmatrix} \\ \begin{pmatrix} p \\ q \end{pmatrix}_{t=0} = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}. \end{cases}$$

Then the algorithm

$$\begin{pmatrix} p_0(t) \\ q_0(t) \end{pmatrix} \xrightarrow{H^n} \begin{pmatrix} p_n(t) \\ q_n(t) \end{pmatrix} \xrightarrow{f} \begin{pmatrix} a_n(t) \\ b_n(t) \end{pmatrix}$$

yields a solution to the Toda lattice equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} a_n \\ b_n \end{pmatrix} = X^{(0)} = \begin{pmatrix} a_n(b_{n+1} - b_n) \\ 2(a_n^2 - a_{n-1}^2) \end{pmatrix}.$$

The discrete evolution behaviour of the numerical solution is shown in figure 1, where $N = 2$ and

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad p_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad q_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \gamma = 10 \qquad \delta_2 = 0.$$

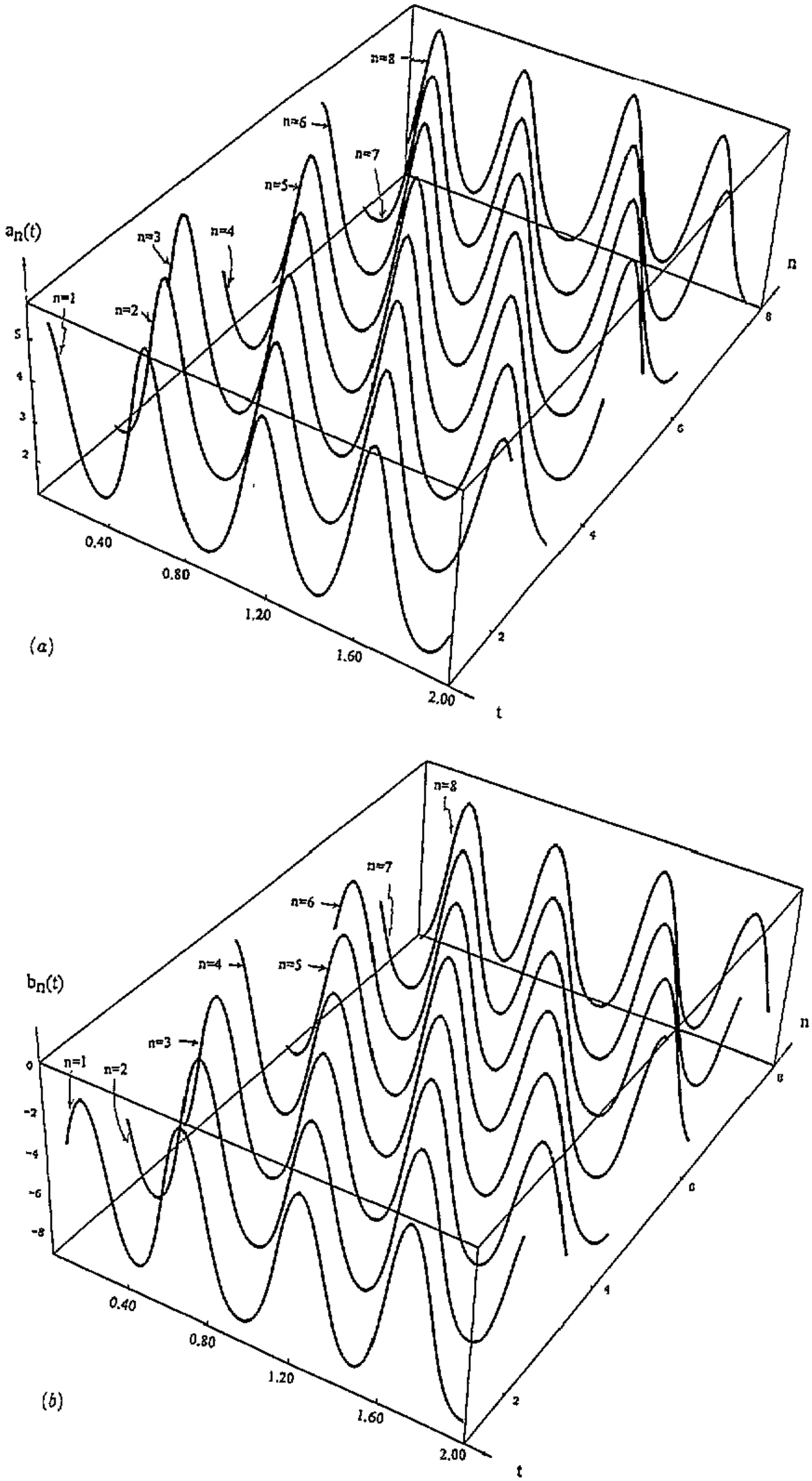

Figure 1. Time evolution of the finite-band solution $(a_n(t), b_n(t))$ of the Toda equation.

7. The discrete Neumann system

$N+1$ distinct eigenparameters are taken instead of N in the Bargmann–Garnier case. Let $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_{N+1})$, $\Phi_n = (\Phi_n^{(1)}, \ldots, \Phi_n^{(N+1)})$. Consider the Neumann system [9]:

$$\begin{cases} a_{n-1}\Phi_{n-1} + b_n\Phi_n + a_n\Phi_{n+1} = \Lambda\Phi_n \\ \begin{pmatrix} a_n^{-1} \\ 1 \end{pmatrix} = \xi_n^{(-1)} = \sum_{j=1}^{N+1} \nabla\lambda_j = \begin{pmatrix} 2\langle\Phi_n, \Phi_{n+1}\rangle \\ \langle\Phi_n, \Phi_n\rangle \end{pmatrix}. \end{cases}$$

By introducing $p_n = a_{n-1}\Phi_{n-1}$, $q_n = \Phi_n$, the Neumann system is equivalent to the symplectic map

$$H: \begin{pmatrix} p_n \\ q_n \end{pmatrix} \to \begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} a_n q_n \\ a_n^{-1}\{(\lambda - b_n)q_n - p_n\} \end{pmatrix}$$

where

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = f(p_n, q_n) = \begin{pmatrix} \|(\Lambda - b_n)q_n - p_n\| \\ \langle\Lambda q_n, q_n\rangle - 1 \end{pmatrix}.$$

The total-difference formula (2.9) provides a natural and systematic way of yielding the conserved integrals of H. Let $P(\lambda) = (\lambda - \lambda_1)\cdots(\lambda - \lambda_{N+1})$ and

$$G^{(N)} = \sum_{j=1}^{N+1} \frac{P(\lambda)}{\lambda - \lambda_j}\nabla\lambda_j = P(\lambda)\begin{pmatrix} 2Q_\lambda(\Phi_n, \Phi_{n+1}) \\ Q_\lambda(\Phi_n, \Phi_n) \end{pmatrix}.$$

It is easy to see that

$$(K - \lambda J)G^{(N)} = 0.$$

Therefore, we obtain the generating function of the integrals for H:

$$\mathcal{F}_\lambda = \frac{1}{2P^2(\lambda)} G^{(N)} \cdot TG^{(N)} = Q_\lambda(p, q) + \begin{vmatrix} Q_\lambda(p, p) & Q_\lambda(p, q) \\ Q_\lambda(q, p) & Q_\lambda(q, q) \end{vmatrix}.$$

The Neumann representation, similar to theorem 4.6 and 4.7, can be established in the phase space TS^N.

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